REFERENCES

1. Feigin, M. I., Doubling of the oscillation period with C-bifurcations in piecewise-continuous systems. PMM Vol. 34, № 5, 1970.
2. Bautin, N. N., Behavior of Dynamic Systems Close to the Boundaries of the Stability Region. Leningrad-Moscow, Gostekhizdat, 1949.
3. Iorish, Iu. I., Subharmonic resonance in a system with an elastic limiter of motion. Zh. Tekhn. Fiz., Vol. 16, № 6, 1946.
4. Bespalova, L. V., On the theory of a vibropercussive mechanism. Izv. Akad. Nauk SSSR, OTN, № 5, 1957.
5. Babitskii, V. I., On the existence of high-frequency oscillations of large amplitude in linear systems with limiters. Mashinovedenie, № 1, 1966.
6. Kolovskii, M. Z., Nonlinear Theory of Vibroprotective Systems. Moscow, "Nauka", 1966.
7. Astashev, V. K., On the dynamics of an oscillator impacting on a limiter. Mashinovedenie, № 2, 1971.
8. Biderman, V. L., Applied Theory of Mechanical Oscillations. Moscow, "Vysshaia Shkola", 1972.

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CONSTRUCTION OF SOLUTIONS OF NONLINEAR TWO-DIMENSIONAL PROBLEMS ON CURRENT DISTRIBUTION IN AN ANISOTROPICALLY CONDUCTING MEDIUM

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Stationary two-dimensional electric current distributions in an anisotropically conducting medium having a nonlinear Ohm's law, are described by the system of equations formulated in [1]. Depending on the character of the nonlinear relation between the current density and the electric field, and on the value of the Hall parameter β, this system can be of an elliptic or hyperbolic type. For $\beta = 0$ the electrodynamic equations are analogous to the equations for potential gas dynamic flows, therefore by analogy these problems can be solved by the hodograph transformation, as it is done in gas dynamics [2]. The hodograph transformation generalized for the case $\beta \neq 0$ is applied below to simple two-dimensional problems. The relation between the type of system and the positive definiteness of the symmetric part of the differential conductivity tensor, is established. Linear equations in the hodograph plane of an effective electric field are obtained for the potential and for a function of the electric current. Boundary conditions are formulated in terms of each of these functions on the image lines for the electrode and dielectric regions with straight-line boundaries. For the elliptic case the solution of two asymptotic problems are obtained and examined: (1) the field distribution in a strip between a perfectly conducting wall and a dielectric wall; (2) the current concentration in the region of a semi-infinite electrode edge. The possibility of corresponding solutions for the hyperbolic case is discussed. For $\beta \neq 0$ exact solutions for particular depen-

dence of the electrical conductivity on current density, corresponding to the hyperbolic region, are obtained by the method of characteristics used in [3, 4]. There are reasons for assuming that the distribution is unstable for hyperbolic modes [1, 3–5]. For homogeneous states of a nonequilibrium plasma, the β value corresponding to a change of the system type, also determines the ionization instability limit. In the general case of nonhomogeneous states the instability of hyperbolic and mixed solutions is not proved. In the elliptic region the equations describe stable current distributions, therefore the construction of elliptic solutions is of the greatest physical interest.

1. Let us consider an incompressible conducting medium moving at velocity $\mathbf{V}(x, y) = V_x\mathbf{e}_x + V_y\mathbf{e}_y$ in a homogeneous magnetic field $\mathbf{B} = B\mathbf{e}_z$, $B = \text{const} > 0$. If the magnetic Reynolds number is small and the current density vector $\mathbf{j}$ lies in the plane xy and is independent of the coordinate z, the system of electrodynamic equations is expressed by

$$\partial j_x / \partial x + \partial j_y / \partial y = 0, \qquad \partial q_x / \partial y - \partial q_y / \partial x = 0 \tag{1.1}$$
$$j_x = (1 + \beta^2)^{-1} \sigma (q_x - \beta q_y), \qquad j_y = (1 + \beta^2)^{-1} \sigma (\beta q_x + q_y)$$
$$(\mathbf{q} = \mathbf{E} + \mathbf{V} \times \mathbf{B})$$

Here $\mathbf{q}$ is the electric field in the concomitant frame of reference. The second equation of (1.1) follows from the condition $\operatorname{rot}(\mathbf{V} \times \mathbf{B}) = 0$. In the general case the electroconductivity σ and the Hall parameter β depend on the current density, and according to Ohm's law the relation between the moduli of the vectors j and q is given by the formula

$$j = [1 + \beta^2(j)]^{-1/2} \sigma(j)\, q \tag{1.2}$$

The type of system (1.1) is determined by the sign of the expression

$$\Delta = \frac{d \ln (j/\sigma) / d \ln j}{[d \ln (\sigma/\beta) / d \ln j]^2} - \frac{\beta^2}{4} = \frac{\lambda d \ln (j/\sigma) / d \ln q}{[d \ln (\sigma/\beta) / d \ln q]^2} - \frac{\beta^2}{4} \tag{1.3}$$
$$(\lambda = d \ln j / d \ln q)$$

For $\Delta > 0$ the system type is elliptic and for $\Delta < 0$ it is a hyperbolic one. In the sequel we shall assume $\lambda > 0$ which corresponds to a monotonically increasing dependence $j(q)$ taking place in many practically interesting cases. Besides, we take $\beta = \text{const}$. This assumption agrees qualitatively with the characteristics of a partly ionized nonequilibrium plasma, the Hall parameter of which is much less sensitive to changes in the current density compared with the dependence of $\sigma(j)$. The application of these assumptions allows the condition $\Delta > 0$ to be set in the following form:

$$\beta < \beta_* = 2\sqrt{\lambda} / |1 - \lambda| \tag{1.4}$$

Here β_* is the critical value of the Hall parameter, beyond which the type of system (1.1) changes. In the case $\beta = 0$ the hyperbolic region corresponds to the range of q or j changes, where $\lambda < 0$, i.e. a negative differential conductivity occurs. For $\lambda > 0$ the volt-ampere characteristic $j(q)$ does not contain falling parts. However, in this case also the hyperbolic character of the system for $\beta > \beta_*$ is related to an onset of negative differential conductivity in wider meaning of this term. The differential conductivity of an anisotropic conducting medium is characterized by the tensor $\boldsymbol{\sigma}_d$

which can be introduced by linearizing the Ohm's law

$$\delta \mathbf{j} = \sigma_d \cdot \delta \mathbf{q}, \quad \| \sigma_d \| = \| \partial j_\alpha / \partial q_\beta \|$$

Let us decompose the tensor σ_d into its symmetric and antisymmetric parts: $\sigma_d = \sigma_s + \sigma_a$. For $\beta = \text{const}$ the matrices of these tensors in the system of the principal axes of the tensor σ_s are of the form

$$\| \sigma_s \| = \Lambda \begin{Vmatrix} 1+H & 0 \\ 0 & 1-H \end{Vmatrix}, \qquad \| \sigma_a \| = \beta\Lambda \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix} \tag{1.5}$$

$$\Lambda = {}^1/_2 (\lambda + 1)\, \sigma / (1+\beta^2), \qquad H = [(1+\beta^2)/(1+\beta_*^2)]^{1/2} \operatorname{sign}(\lambda - 1)$$

The requirement for the tensor σ_s to be positive-definite is expressed by $H^2 < 1$ and is in agreement with the condition (1.3). The principal axes of the tensor σ_s are directed along the bisector of the angle formed by vectors $\mathbf{q}$ and $\mathbf{j}$ and the normal to this bisector. In the case of a linear medium $H = 0$ and the tensor σ_s is isotropic. Let $\mathbf{l}$ be an arbitrary direction in the physical plane. Then the derivative $\partial j_l / \partial q_l$ has the form

$$\partial j_l / \partial q_l = \mathbf{l}(\sigma_d \cdot \mathbf{l}) = \mathbf{l}(\sigma_s \cdot \mathbf{l})$$

Here use is made of the fact that the convolution $\mathbf{l}(\sigma_a \cdot \mathbf{l})$ vanishes. It follows from the results obtained that the elliptic character of the system (1.1) is equivalent to a monotonic increase of j_l with increase of q_l for all directions of $\mathbf{l}$.

Let $\mathbf{e}_i^*$ be the basis of the principal axes of the tensor σ_s, so the vector $\mathbf{e}_1^*$ bisects the angle between $\mathbf{q}$ and $\mathbf{j}$. We represent the unit vector $\mathbf{l}$ in the form

$$\mathbf{l} = \cos\chi\, \mathbf{e}_1^* + \sin\chi\, \mathbf{e}_2^*$$

Then the condition $\partial j_l / \partial q_l > 0$ for $\beta > \beta_*$ will be violated in the following regions of variation of the χ-angle:

$$| \chi | < {}^1/_2\, \omega, \quad | \chi - \pi | < {}^1/_2 \omega \qquad 0 < \lambda < 1$$

$$| \chi - {}^1/_2 \pi | < {}^1/_2\, \omega, \quad | \chi - {}^3/_2 \pi | < {}^1/_2\, \omega \qquad (\lambda > 1)$$

$$0 < \omega = \arccos (1 / |H|) < \theta = \operatorname{arc\,tg} \beta$$

If $\mathbf{l}$ and $\mathbf{m}$ are orthogonal vectors, then the Jacobian $\partial (j_l, j_m) / \partial (q_l, q_m)$ is always positive for $\lambda > 0$ and the vectors $\mathbf{j}$ and $\mathbf{q}$ possess one-to-one relation. At the same time the Jacobian

$$\partial (j_l, q_m) / \partial (j_m, q_l) = [(1 + H \cos 2\chi) / (1 - H \cos 2\chi)]$$

for $H^2 > 1$ can change sign, because the angle χ can assume arbitrary values as the direction of the vector $\mathbf{q}$ changes. Hence for $\beta > \beta_*$ there exists a region of values of the vector $\mathbf{q}$, for which it is impossible to define one pair of the variables (j_l, q_m) and (j_m, q_l) by the other. As an example, the curves of Fig. 1 illustrate the dependence of the value $j_x^* = j_x(q_x, q_y) / j_y(0, q_y)$ on $q_x^* = q_x / q_y$ for $\lambda \equiv 2$ and different values of β. For $\beta > \beta_* = 2\sqrt{2}$ a falling part appears on the curves and the line $j_x^* = \text{const}$ can cross the curve of the function $j_x^*(q_x^*)$ at more than one point. In this case, at least two sets of values j_y, q_x correspond to each of fixed values j_x, q_y

We shall indicate one more result of the above discussion. If for $\beta > \beta_*$ a constant component of current density j_x and constant component q_y in the ambiguity region of the mapping of $(j_x, q_y) \to (j_y, q_x)$, are specified, it is possible to construct formally discontinuous solutions in the form of layers, bounded by the $x = \mathrm{const}$ planes; the distribution of the vectors $\mathbf{q}$ and $\mathbf{j}$ will be uniform inside each layer. On the common boundaries of the layers, the values q_x and j_y will undergo a jump, while the values of q_y and j_x remain continuous, according to the conventional electrodynamic conditions for the surfaces of dicontinuity. A hypothesis was suggested in [6] according to which a development of the ionization instability in a nonequilibrium plasma results in the onset of piecewise-homogeneous structures in this plasma.

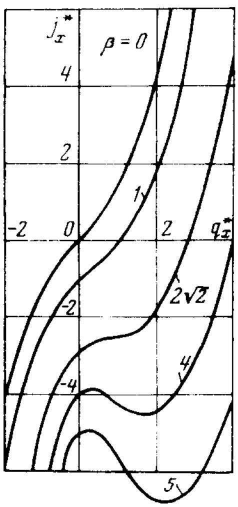

Fig. 1

2. On the basis of the first two equations (1.1) we introduce the stream function ψ and the potential φ, so that

$$\mathbf{j} = \mathrm{rot}\,(\psi \mathbf{e}_z), \qquad \mathbf{q} = -\nabla\varphi \tag{2.1}$$

Let α and γ be the angles formed by the positive direction of the x-axis with the vectors $\mathbf{q}$ and $\mathbf{j}$, respectively. These angles are connected by the relation

$$\gamma - \alpha = \theta = \mathrm{arc\,tg}\,\beta \tag{2.2}$$

According to [7], for the case $\beta = \mathrm{const}$ we write the first two equations from (1.1) in orthogonal curvilinear coordinates associated with the lines of the vector $\mathbf{q}$

$$\begin{gathered} \partial j / \partial s - \beta j \partial\alpha / \partial s + j \partial\alpha / \partial n + \beta \partial j / \partial n = 0 \\ \partial q / \partial n - q \partial\alpha / \partial s = 0 \end{gathered} \tag{2.3}$$

The operators $\partial / \partial s$ and $\partial / \partial n$ indicate differentiation in the direction of the vector $\mathbf{q}$ and of its normal, respectively. Equations (2.1) are equivalent to the relations

$$\partial\varphi / \partial s = -q, \quad \partial\varphi / \partial n = 0, \quad \partial\psi / \partial s = -j \sin\theta, \quad \partial\psi / \partial n = j \cos\theta$$

From these relations we obtain the following formulas:

$$\begin{aligned} &\partial q / \partial s = J\,(q \partial\psi / \partial\alpha - j \sin\theta \partial\varphi / \partial\alpha), \quad \partial q / \partial n = J j \cos\theta \partial\varphi / \partial\alpha \\ &\partial\alpha / \partial s = J\,(j \sin\theta\, \partial\varphi / \partial q - q \partial\psi / \partial q) \\ &\partial\alpha / \partial n = -J j \cos\theta \partial\varphi / \partial q \\ &J = \partial\,(q, \alpha) / \partial\,(\psi, \varphi) \end{aligned} \tag{2.4}$$

Assuming $J \neq 0$, after substituting (2.4) into (2.3) and eliminating J, we obtain the following linear system (*) :

$$\begin{gathered} \partial\varphi / \partial q - \cos\theta\, [(\beta q / j)\, \partial\psi / \partial q + (\lambda / j)\, \partial\psi / \partial\alpha] = 0 \\ \beta q\, \partial\varphi / \partial q - \partial\varphi / \partial\alpha - (q^2 / j \cos\theta)\, \partial\psi / \partial q = 0 \end{gathered} \tag{2.5}$$

Since $\theta = \mathrm{const}$, we can replace the operator $\partial / \partial\alpha$ by $\partial / \partial\gamma$. Further we use the variables of a "mixed" hodograph q, γ. We eliminate in turn the functions φ and ψ from (2.5) and obtain the second order linear equations

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*) Linear system of equations in the plane of the independent variables j, γ is examined in [2].

$$L(\psi) + (2 - \lambda) q^{-1} \partial\psi / \partial q + \beta\lambda (\lambda - 1 - \mu) q^{-2} \partial\psi / \partial\gamma = 0 \quad (2.6)$$
$$L(\varphi) + (\lambda - \mu) q^{-1} \partial\varphi / \partial q + \beta (\lambda - 1 - \mu) q^{-2} \partial\varphi / \partial\gamma = 0$$
$$L \equiv \frac{\partial^2}{\partial q^2} + \frac{\beta(1-\lambda)}{q} \frac{\partial^2}{\partial q\, \partial\gamma} + \frac{\lambda}{q^2} \frac{\partial^2}{\partial\gamma^2}, \quad \lambda = \frac{d \ln j}{d \ln q}, \quad \mu = \frac{d \ln \lambda}{d \ln q}$$

For solving boundary value problems, the system (2.5) or any equation from (2.6) can be taken as a basis. If Γ_1 and Γ_2 are the images on the plane $q\gamma$ of the electrode and dielectric regions boundary, respectively, then

$$(\varphi)_{\Gamma_1} = \text{const}, \qquad (\psi)_{\Gamma_2} = \text{const} \qquad (2.7)$$

The first of the conditions (2.7) must be satisfied by the potential of the electric field $\mathbf{E}$ in a fixed frame of reference. But it will be satisfied also for a potential of the effective field $\mathbf{q}$, if the normal velocity component on the electrode wall equals zero.

In a general case the lines Γ_1 and Γ_2 are unknown. Moreover, the boundary conditions (2.7) contain different functions which make the problems formulation more difficult when one of Eqs. (2.6) is taken as a basis.

The first of these difficulties is partly, and the second one fully removed, if the boundary region in the physical plane is formed by straight-line segments. On such segments $\gamma = \text{const}$ and therefore Γ_1 and Γ_2 consist of segments which are parallel to the q-axis. But the positions of the ends of the segments are not always known. For a straight-line electrode and dielectric the boundary conditions can be formulated in terms of the functions ψ and φ, respectively. For this purpose the relations (2.4) and conditions

$$(\partial\alpha / \partial n)_{\Gamma_1^*} = 0, \qquad (\cos\theta\, \partial\alpha / \partial s + \sin\theta\, \partial\alpha / \partial n)_{\Gamma_2^*} = 0$$

should be used on the electrode $\Gamma_1{*}$ and the dielectric $\Gamma_2{*}$ in the physical plane. As a result we obtain

$$(\lambda \partial\psi / \partial\gamma + \beta q \partial\psi / \partial q)_{\Gamma_1} = 0, \qquad (\partial\varphi / \partial\gamma - \beta q \partial\varphi / \partial q)_{\Gamma_2} = 0 \quad (2.8)$$

When the solution of the problem for the function ψ (consequently, also for φ) is known, then the transition to the physical plane is accomplished by means of the equations

$$\frac{\partial \mathbf{r}}{\partial q} = K\left(\mathbf{j} \frac{\partial\varphi}{\partial q} + \mathbf{q}_* \frac{\partial\psi}{\partial q}\right), \qquad \frac{\partial \mathbf{r}}{\partial\gamma} = K\left(\mathbf{j} \frac{\partial\varphi}{\partial\gamma} + \mathbf{q}_* \frac{\partial\psi}{\partial\gamma}\right) \qquad (2.9)$$
$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y, \quad \mathbf{q}_* = q_y \mathbf{e}_x - q_x \mathbf{e}_y$$
$$K = \partial(x, y) / \partial(\varphi, \psi) = -\sigma / j^2$$

The function $\mathbf{r}(\mathbf{q})$ can be found from the system (2.9), using the method of quadratures. The inverse function $\mathbf{q}(\mathbf{r})$ is single-valued, if the following Jacobian is nonzero:

$$D = \frac{\partial(x, y)}{\partial(q, \gamma)} = -\frac{\sigma \cos\theta}{j^3} \left[q^2 \left(\frac{\partial\psi}{\partial q}\right)^2 + \beta(1-\lambda) q \frac{\partial\psi}{\partial q} \frac{\partial\psi}{\partial\gamma} + \lambda \left(\frac{\partial\psi}{\partial\gamma}\right)^2 \right] \quad (2.10)$$

The quadratic form on the right-hand side of (2.10) is a fixed sign form if $\beta < \beta_*$. Therefore for elliptic solutions the Jacobian can vanish only at isolated points. For $\beta \geqslant \beta_*$ in the physical plane a line on which $D = 0$, can exist. Using gas dynamics terminology such a line can be called the limiting line [7]. In the vicinity of the limiting line it is impossible to construct a single-valued and continuous distribution $q(\mathbf{r})$. The condition $\beta \geqslant \beta_*$ is necessary but not sufficient for the existence of a limiting line. Adequate examples of the hyperbolic versions are examined below.

Further, the following model dependence is considered:

$$j(q) = Aq^{\lambda} \quad (A = \text{const} > 0, \quad \lambda = d \ln j / d \ln q = \text{const} > 0) \qquad (2.11)$$

For $\lambda > 1$ this dependence corresponds to the behavior of a low-temperature plasma in the state of a nonequilibrium ionization (excluding the weak current region where, in fact, nonlinear effects do not occur). For $\lambda = \text{const}$ in Eqs. (2.6) $\mu = 0$ should be assumed.

3. One of the simple two-dimensional current distributions is described by the function

$$\psi(q, \gamma) = Cq^{\lambda} \sin \gamma, \qquad C = \text{const} < 0 \qquad (3.1)$$

The function (3.1) is examined in the half-strip

$$0 < q < \infty, \qquad 0 \leqslant \gamma \leqslant \gamma_m = {}^1/_2\pi + \theta \qquad (3.2)$$

and satisfies the first equation of (2.6) and the boundary conditions

$$\psi(0, \gamma) = 0, \quad \psi(q, 0) = 0, \quad (\lambda \partial\psi / \partial\gamma + \beta q \partial\psi / \partial q)_{\gamma=\gamma_m} = 0$$

In the physical plane the solution (3.1) describes the current flow between the electrode $(y = -\delta)$ and the dielectric $(y = 0)$ walls, in a strip of a constant width $(-\delta \leqslant y \leqslant 0, \ |x| < \infty)$. The sign of the constant C is chosen to satisfy the condition $j \to 0$ for $x \to -\infty$. The current lines leaving the electrode at an angle $\gamma = \gamma_m$ spread at some distance from it and continue to infinity on the right, coming into contact with the dielectric wall as $x \to \infty$.

By analogy with corresponding linear problems for $\beta < \beta_*$, it can be expected that this solution in the region $-\infty < x \lessapprox -\delta$ will define the resulting asymptotics for a channel with the displaced semi-infinite electrodes $y = -\delta, x < 0$ and $y = 0, x > 0$. The estimation of the dependence of the characterisitc lengths of the falling parts of the current and field as $x \to -\infty$, relative to the channel dimensions, on the parameters λ and β, is of particular interest.

By integrating Eq. (2.9) we find the functions $x(q, \gamma)$ and $y(q, \gamma)$ to within the additive constants. The constant in the expression for $y(q, \gamma)$ and the value C from (3.1) are found from the conditions $y(q, 0) = 0$, $y(q, \gamma_m) = -\delta$. The constant in the expression for $x(q, \gamma)$ can be determined by assuming, for example, that the entire current I flows through the electrode to the left of the point $x = 0$, $y = -\delta$. Finally we obtain the following formulas:

$$x = h\ \{\lambda \ln (q / q_0) + {}^1/_4 (\lambda - 1) [\cos (2\gamma - \theta) / \cos \theta + 1 + 2\beta (\gamma - \gamma_m)]\} \qquad (3.3)$$

$$y = \delta \gamma_m^{-1} \{{}^1/_2 H [\sin (2\gamma - \theta) + \sin \theta] - \gamma\}$$

$$h = 2\delta / [(\lambda + 1)\gamma_m], \qquad q_0 = (I / Ah \cos \theta)^{1/\lambda}$$

The value H is defined in (1.5). From the relations (3.3) and (2.11) we find the distribution of the effective field, the current density and the local dissipation on the electrode

$$q = q_0 \exp(x / h_q), \quad j = Aq_0^{\lambda} \exp(x / h_j) \tag{3.4}$$
$$g = j^2 / \sigma = Aq_0^{\lambda+1} \cos\theta \exp(x / h_g)$$
$$h_q / \delta = 2\gamma_m^{-1} \lambda / (\lambda + 1), \quad h_j / \delta = 2\gamma_m^{-1} / (\lambda + 1)$$
$$h_g / \delta = 2\gamma_m^{-1} \lambda / (\lambda + 1)^2$$

It follows from (3.4) that the length of the falling part of the effective field h_q will be a monotonically increasing, and the length of the falling part of the current density h_j will be a decreasing function of λ. The length of the decay of dissipation h_g reaches its maximum value when $\lambda = 1$, i.e. in the case of linear medium. Increasing the Hall parameter causes a decrease (by factor not greater than two) of all lengths mentioned above.

Let us consider the Jacobian D of the expression (3.3)

$$D = \partial(x, y) / \partial(q, \gamma) = {}^1/_2 h^2 \lambda (\lambda + 1) q^{-1} [H \cos(2\gamma - \theta) - 1]$$

In the case $\beta \geqslant \beta_*$ we have $H^2 \geqslant 1$, and the value D vanishes in the interval $(0, \gamma_m)$ at the points $\gamma = {}^1/_2 (\theta \pm \omega)$ for $H \geqslant 1$ or at the points $\gamma = {}^1/_2 (\pi + \theta \pm \omega)$ for $H \leqslant -1$, where $\omega = \arccos(1 / |H|)$. Limiting lines $y = \text{const}$, at least one of which lies inside the strip $-\delta < y < 0$, correspond to the indicated values of γ in the physical plane. As in gas dynamics [7] the limiting lines form an envelope for one family of characteristics. The function $y = y(\gamma)$ determined in (3.3) has two extrema in the interval $(0, \gamma_m)$. Therefore an inverse function $\gamma(y)$ continuous on the segment $[-\delta, 0]$ and satisfying the conditions $\gamma(-\delta) = \gamma_m$, $\gamma(0) = 0$, does not exist. Thus, in the hyperbolic region the above problem has no continuous solution. In the parabolic case $(H^2 = 1)$ a continuous inverse function $\gamma(y)$ exists, and current lines have infinite curvature at the limiting line points.

We shall note the influence of nonlinear effects on the character of the current spreading in the ellipticity region. With increase in the parameter λ the decrease of the function $\gamma(y)$ slows down near the electrode and intensifies near the dielectric. At the same time, the zone which is adjacent to the dielectric and in which $j_x > 0$, diminishes.

4. In examining different applied problems, one of the important questions is to determine the asymptotics and qualitative pattern of the field near the electrode edge. To explain this question let us examine a model problem on the current distribution in the half-plane $y \geqslant 0$, the boundary of which consists of an electrode $(y = 0, x < 0)$ and an insulator $(y = 0, x > 0)$, both being semi-infinite. The corresponding solution for the case of an isotropic nonlinear medium is given in [8].

As in the problem from Sect. 3, the region in the hodograph plane represents a half-strip determined by the inequality (3.2). The boundary conditions for the current function have the form:

$$\psi(\infty, \gamma) = 0, \quad \psi(q, 0) = 0, \quad (\lambda \partial\psi / \partial\gamma + \beta q \partial\psi / \partial q)_{\gamma=\gamma_m} = 0 \tag{4.1}$$

The first condition of (4.1) is based on the assumption of a finite total current flowing through any section of the electrode $(x, 0)$ and of a conversion of the value q into infinity at the end point. The structure of the first equation of (2.6) for $\mu = 0$ and the form of the boundary conditions (4.1) permit a solution to be found in the form

$$\psi(q,\gamma) = q^{\varkappa} f(\gamma), \quad \varkappa = \mathrm{const} < 0 \tag{4.2}$$

The constant $\varkappa$ is to be determined later. The function $f(\gamma)$ must be the solution of a homogeneous boundary value problem

$$\lambda f'' + \beta(1-\lambda)(\varkappa-\lambda) f' + \varkappa(1+\varkappa-\lambda) f = 0 \tag{4.3}$$
$$f(0) = 0, \quad \lambda f'(\gamma_m) + \varkappa\beta f(\gamma_m) = 0$$

On the basis of physical understqnding, only those solutions of the problem (4.3) should be considered which correspond to the negative eigenvalues of $\varkappa$. Depending on the parameters the solution can assume one of the following forms:

$$f(\gamma) = Ce^{a\gamma}\sin u\gamma \quad (d>0),\ f(\gamma) = Ce^{a\gamma}\,\mathrm{sh}\, v\gamma \quad (d<0) \tag{4.4}$$
$$a = {}^1/_2\, \beta(\lambda-1)(\varkappa-\lambda)/\lambda, \quad d = \varkappa(1+\varkappa-\lambda)/\lambda - a^2,$$
$$u = \sqrt{d}, \quad v = \sqrt{-d}$$

If $d = 0$, then $f = C\gamma \exp a\gamma$. To determine the values of $\varkappa$ it is necessary to use the last boundary condition of (4.3) which is reduced to the form

$$\lambda u \,\mathrm{ctg}\, \gamma_m u = -(\lambda a + \varkappa\beta) \quad (d>0) \tag{4.5}$$
$$\lambda v \,\mathrm{cth}\, \gamma_m v = -(\lambda a + \varkappa\beta) \quad (d<0)$$

In fact, it is more convenient to consider the relations (4.5) as transcendental equations with respect to u and v, assuming $\varkappa = \varkappa(u)$ or $\varkappa = \varkappa(v)$. The latter dependences are determined by the formulas

$$\varkappa(u;\ \lambda,\ \beta) = (\lambda - 1 - 2\lambda\xi - \sqrt{r(u^2)})\,/\,[2(1-\xi)] \tag{4.6}$$
$$\varkappa(v;\ \lambda,\ \beta) = (\lambda - 1 - 2\lambda\xi \mp \sqrt{r(-v^2)})\,/\,[2(1-\xi)]$$
$$\xi = \beta^2/\beta_*^2, \quad r(u^2) = (\lambda - 1 - 2\lambda\xi)^2 + 4\lambda(1-\xi)(\lambda\xi + u^2)$$

The positive sign in front of the radical in the formula for $\varkappa(v)$ is taken for $\xi > \xi'(\lambda)$, where $\xi' < 1$ is the value of the parameter ξ for which the solution v of the second equation (4.5) causes the function $r(-v^2)$ to vanish. Equations (4.5) are invariant with respect to a change in sign of the values u and v which is equivalent to the change in sign of the constant C. Therefore, without restriction of generality we can assume $u \geqslant 0$, $v \geqslant 0$. For $0 \leqslant \beta \leqslant \beta_1$ the first equation of (4.5) has a discrete set of such solutions u_k that

$$\gamma_m u_k \in (k\pi,\ {}^1/_2\pi + k\pi), \qquad k = 0,\ 1,\ 2,\ \ldots$$

The relation beteeen the values β_1 and β_* has the form

$$\beta_*(\beta_1) = [(1+\beta_1^2)(1+\beta_1\gamma_m(\beta_1))^2 - 1]^{1/2} > \beta_1$$

The roots u_k for $k \geqslant 1$ must be discarded as they do not correspond to the initial statement of the problem. If the values u were to coincide with one of these roots, then the function $f(\gamma)$ would vanish for $\gamma = \pi/u < \gamma_m$. Hence an inclined straight-line insulated wall would exist in the physical plane and this is not assumed in the statement of the problem.

With β increasing from 0 to β_1 the root u_0 of the first equation (4.5) decreases monotonously from 1 to 0. For $\beta > \beta_1$ the second equation of (4.5) should be examined to prove its single-valued solvability. The root v of this equation increases monotonously with increase of β, running along the straight half-line $(0, \infty)$ with change of β from β_1 to ∞. We note that the passage through the values $\beta = \beta_*$ occurs in a continuous way, because the function $\varkappa(v; \lambda, \beta)$ is regular with respect to the parameter β for $\beta = \beta_*$.

From the relations (4.5) it is possible to find the dependence $\beta_*(\beta, u)$ or $\beta_*(\beta, v)$ in an explicit form. Therefore performing the calculations it is convenient to construct lines $u = \text{const}$ and $v = \text{const}$ on the plane $\lambda\beta$. The results of this construction are shown

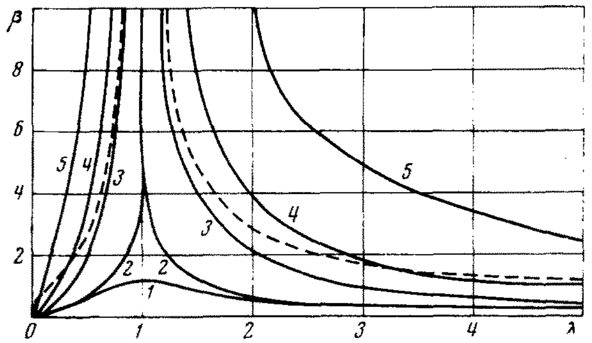

Fig. 2

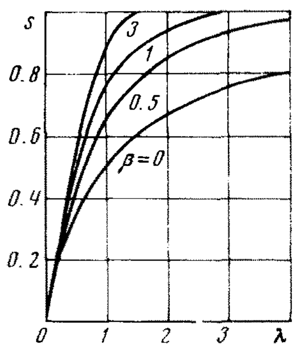

Fig. 3

in Fig. 2. The curves 1 – 5 correspond to the following values of u, v:

$1 - u = 0.3$, $2 - u = v = 0$, $(\beta = \beta_1)$; $3 - v = 1$, $4 - v = 2$, $5 - v = 5$.

The dotted line shows the dependence $\beta_*(\lambda)$.

By integrating Eq. (2.9) we obtain the mapping of the hodograph variables on the physical plane. The integration constants are determined by the conditions $x = y = 0$ for $q = \infty$. The constant C from (4.4) is fixed by specifying the integral current I flowing through the electrode end section, the length h of which is given.

The final formulas for $\beta < \beta_1$ are:

$$x = -G(q/q_h)^{\varkappa-\lambda} e^{a(\gamma-\gamma_m)} [\eta_1(\gamma)\sin u\gamma + \eta_2(\gamma)\cos u\gamma] \qquad (4.7)$$

$$y = G(q/q_h)^{\varkappa-\lambda} e^{a(\gamma-\gamma_m)} [\eta_1'(\gamma)\sin u\gamma + \eta_2'(\gamma)\cos u\gamma]$$

$$G = \frac{h\sin\gamma_m}{\varkappa \sin u\gamma_m}, \quad q_h = \left[\frac{I}{(1-\lambda/\varkappa)Ah\cos\theta}\right]^{1/\lambda}$$

$$\eta_1 = \varkappa\sin\gamma + \lambda a\cos\gamma, \quad \eta_2 = \lambda u\cos\gamma$$

Primes at the functions $\eta_{1,2}$ denote differentiation with respect to γ. In the formulas

(4.7) for $\beta > \beta_1$ the following substitution must be made:

$$\sin u\gamma_m \to \operatorname{sh} v\gamma_m, \quad \sin u\gamma \to \operatorname{sh} v\gamma, \quad \cos u\gamma \to \operatorname{ch} v\gamma$$
$$\eta_2 \to \lambda v \cos \gamma$$

and for $\beta = \beta_1$ – the substitution

$$\sin u\gamma_m \to \gamma_m, \quad \sin u\gamma \to \gamma, \quad \cos u\gamma \to 1, \quad \eta_2 \to \lambda \cos \gamma$$

The distribution of the values q, j and $g = j^2 / \sigma$ on the electrode surface is described by the power functions in the following form:

$$q = q_h x_*^{-p}, \quad j = j_h x_*^{-s}, \quad g = q_h j_h x_*^{-(s+p)} \cos \theta \tag{4.8}$$
$$x_* = -x / h, \quad j_h = A q_h^{\lambda}, \quad p = 1 / (\lambda - \varkappa), \quad s = \lambda / (\lambda - \varkappa)$$

An examination shows that there are inequalities

$$\varkappa < 0 \quad (\lambda \geqslant 1), \quad \varkappa < -1 + \lambda \quad (0 < \lambda < 1)$$

These inequalities are equivalent to the conditions $0 < p < 1$, $0 < s < 1$ which guarantee convergence of the integrals of the values q and j with respect to the electrode end section, and of the value g with respect to the bounded two-dimensional region, adjacent to the compound wall.

In Fig. 3 a family of curves depicts the dependence of the value s on the parameter λ for different values of β. For $\lambda > 1$ the current concentration in the end part of the electrode increases in comparison with the case of $\sigma = \text{const}$. This assertion agrees with the results obtained by numerical calculations [1]. For $\lambda < 1$ nonlinear phenomena lead to a reduced current concentration. An increase of the Hall parameter causes an increased concentration in both linear [9] and nonlinear cases.

It follows from the formulas (4.7) that the ratio y / x depends only on γ. Consequently, the current lines form a family of similar curves with the homothetic center at the coordinate origin. The influence of nonlinearity on the way the current lines develop is analogous to that determined [8] for an isotropic medium.

The solution obtained exists in the whole upper half-plane $y \geqslant 0$ for any value of β. Particularly, for $\beta \geqslant \beta_*$ the Jacobian D has the form

$$D = \frac{\partial (x, y)}{\partial (q, \gamma)} = \frac{h^2 (\varkappa - \lambda) \sin^2 \gamma_m}{q_h \varkappa^2 \operatorname{sh}^2 v\gamma_m} \left(\frac{q}{q_h}\right)^{2(\varkappa-\lambda)-1} \exp\left[2a (\gamma - \gamma_m)\right] F(\gamma)$$

$$F(\gamma) = \lambda^2 (a \operatorname{sh} v\gamma + v \operatorname{ch} v\gamma)^2 + \varkappa^2 \operatorname{sh}^2 v\gamma - \lambda v^2 \varkappa$$

Since $\varkappa < 0$, then $F(\gamma)$ does not vanish anywhere. Consequently, the functions $q(x, y)$ and $\gamma(x, y)$ are single-valued and continuous. Hence, unlike in the problem examined in Sect. 3, the initial region of the physical plane does not contain limiting lines.

The hyperbolic solution considered represents, in fact, an analytic continuation of the elliptic solution in the parameter β governing the type of a system. The current function determined in this manner is regular with respect to the hodograph variables. The solution of correctly stated hyperbolic problems need not be smoother. At the same time, for incorrect problems (including those for $\beta > \beta_*$) considered above) narrowing of the class of admissible solutions may represent one of methods of regularization [10].

The question of the physical realization of the solution found for $\beta > \beta_*$ is closely connected with the stability problem of the nonuniform current distribution for supercritical values of the Hall parameter.

REFERENCES

1. Oliver, D. A. and Mitchner, M., Nonuniform electrical conduction in MHD channels. AIAA Jounal, Vol. 5, № 8, 1967.

2. Emets, Iu. P., The hodograph method in electrodynamics of continuous nonlinearly conducting media. PMM Vol. 31, № 6, 1967.

3. Glushkov, I. S., Karpukhin, V. T. and Nedospasov, A. V., Current distribution in an $E \times H$ discharge with sectional electrodes for nonequilibrium plasma conductivity. Teplofiz. vysokikh temp., № 2, 1969.

4. Lengyel, L. L., Current and potential distribution in nonequilibrium MHD plasmas at high magnetic field strengths. AIAA Journal, Vol. 9, № 10, 1971.

5. Kulikovskii, A. G. and Regirer, S. A., On stability and evolution of the electric current distribution in a medium with nonlinear conductivity. PMM Vol. 32, № 4, 1968.

6. Dykhne, A. M., Nonlinear phenomena in a plasma at the presence of an ionization instability. Proc. of 5th Internat. Conf. Magnetohydrodynamic Electr. Power Generation. Munich, Vol. 2, 1971.

7. von Mises, R., Mathematical Theory of Compressible Fluid Flows. Moscow, Izd. inostr. lit., 1961.

8. Rutkevich, I. M., A two-dimensional electrodynamic problem of stationary currents in a nonlinearly conducting medium. PMTF, № 6, 1970.

9. Hurwitz, H., Jr., Kilb, R. B. and Sutton, G. N., Influence of tensor conductivity on current distribution in a MHD generator. J. Appl. Phys., Vol. 32, № 2, 1961.

10. Lattes, R. and Lions, J. L., The Method of Quasi-reversibility; Applications to Partial Differential Equations. N. Y., American Elsevier Publ. Co., 1969.

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